ON THE LIPSCHITZ STABILITY OF INVERSE NODAL PROBLEM FOR DIRAC EQUATION

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Abstract: Inverse nodal problem on Dirac operator is finding the parameters in the boundary conditions, the number m and the potential function V in the Dirac equations by using a set of nodal points of a component of two component vector eigenfunctions as the given spectral data. In this study, we solve a stability problem using nodal set of vector eigenfunctions and show that space of all V functions is homeomorphic to the partition set of all space of asymptotically equivalent nodal sequences induced by an equivalence relation. Furthermore, we give a reconstruction formula for the potential function as a limit of a sequence of functions and associated nodal data of one component of vector eigenfunctions. Our method depends on the explicit asymptotic expressions of the nodal points and nodal lengths and basically, this method is similar to [1] and [2] which is given for Sturm-Liouville and Hill operators, respectively.

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1. Introduction

Inverse spectral problems have been an important research issue in mathematical physics. Using spectral data, different methods have been proposed for recovering coefficient functions in differential equations [3], [4], [5], [6], [7], [8], [9], [10]. Normally, the spectral data contains the eigenvalues and a second set of data. In 1988, McLaughlin show that knowledge of the nodal points can determine the potential function of the Sturm-Liouville problem up to a constant [11]. This is so called inverse nodal problem. Numerical schemes were then given by Hald and McLaughlin [12] for reconstruction of the density function of a vibrating string, the elastic modulus of a vibrating rod, the potential function in the Sturm-Liouville problem and the impedance in impedance equation. Independently, Shen and his coworkers studied the relation between the nodal points and density function of string equation [13]. Inverse nodal problems have been studied by many authors [14], [15].

The Dirac equation is a modern presentation of the relativistic quantum mechanics of electrons intended to make new mathematical results accesible to a wider audience. It treats in some depth relativistic invariance of a quantum theory, self-adjointness and spectral theory, qualitative features of relativistic bound and scattering states and the external field problem in quantum electrodynamics, without neglecting the interpretational difficulties and limitations of the theory [16].

Inverse problems for Dirac system had been investigated by Moses [17], Prats and Toll [18], Verde [19], Gasymov and Levitan [20], and Panakhov [21]. It is well known that two spectra uniquely determine the matrix valued potential function [22]. In [23], eigenfunction expansions for one dimensional Dirac operators describing the motion of a particle in quantum mechanics are

investigated. In addition, inverse spectral problems for weighted Dirac systems are considered in [24].

For the Dirac system with a spectral parameter in the boundary conditions, one studied the properties of the eigenvalues and vector-valued eigenfunctions for the Dirac system with the same spectral parameter in the equations and the boundary conditions [25]. The sampling theory of signal analysis associated with Dirac systems when the eigenvalue parameter appears linearly in the boundary conditions was investigated [26]. One investigated a problem for the Dirac differential operators in the case where an eigenparameter not only appears in the differential equation but is also linearly contained in a boundary condition, and proved uniqueness theorems for the inverse spectral problem with known collection of eigenvalues and normalizing constants or two spectra [27].

Inverse nodal problems for Dirac systems had not been studied until the works of Yang, Huang [28]. They give reconstruction formulas for one dimensional Dirac operator by nodal datas. Later years, some problems were solved for inverse nodal problem of Dirac systems [16], [29].

The organization of this paper is as follows: in section 1, we give some properties of Dirac system and quote some important results to use in main theorem. In section 2, we obtain some reconstruction formulas for potential function. Finally, we define d_0, $d_{\Sigma_{Dir}}$, Lipschitz stability and we show $d_{\Sigma_{Dir}}$ is a pseudometric on Σ_{Dir}. Then, we prove Theorem 3.1 in section 3.

Consider the following Dirac system

$$By'(x)+Q(x)y(x)=\lambda y(x), 0\leq x\leq \pi, \tag{1.1}$$

with the boundary conditions

$$\begin{aligned}(\lambda\cos\alpha+a_0)\,y_1(0)+(\lambda\sin\alpha+b_0)\,y_2(0) &= 0,\\ (\lambda\cos\beta+a_1)\,y_1(\pi)+(\lambda\sin\beta+b_1)\,y_2(\pi) &= 0,\end{aligned} \tag{1.2}$$

where

$$B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad Q(x)=\begin{pmatrix}V(x)+m & 0\\ 0 & V(x)-m\end{pmatrix},\quad y(x)=\begin{pmatrix}y_1(x)\\ y_2(x)\end{pmatrix}, \tag{1.3}$$

and V is a real valued, continuous function from the class $L_1[0,\pi]$. $m, a_k, b_k (k=0,1), \alpha$ and β are real constants: moreover $-\frac{\pi}{2}\leq\alpha\leq\frac{\pi}{2}$ and $-\frac{\pi}{2}\leq\beta\leq\frac{\pi}{2}$ [29].

The properties of the eigenvalues and eigenfunctions of the boundary value problem (1.1) were studied in [25]. Throughout their paper, Yang and Pivovarchik assume that

$$\begin{aligned}a_0\sin\alpha-b_0\cos\alpha &> 0,\\ a_1\sin\beta-b_1\cos\beta &< 0.\end{aligned} \tag{1.4}$$

Under this condition, the eigenvalues of the boundary value problem (1.1)-(1.2) are real and algebraically simple [29].

Considering (1.3) in (1.1), we get

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y_1'(x) \\ y_2'(x) \end{pmatrix}+\begin{pmatrix} V(x)+m & 0 \\ 0 & V(x)-m \end{pmatrix}\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}=\lambda\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

and thus, equation (1.1) is equivalent to a system of two simultaneous first order differential equations

$$\begin{aligned} y_2'(x,\lambda)+[V(x)+m]\,y_1(x,\lambda)-\lambda y_1(x,\lambda) &= 0, \\ -y_1'(x,\lambda)+[V(x)-m]\,y_2(x,\lambda)-\lambda y_2(x,\lambda) &= 0. \end{aligned} \tag{1.5}$$

In generally, potential function of Dirac equation (1.1) has the following form

$$Q(x)=\begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix},$$

where $p_{ik}(x)$ $(i,k=1,2)$ are real functions which are defined and continuous on the interval $[0,\pi]$. For the case in which $p_{12}(x)=p_{21}(x)=0$ and $p_{11}(x)=V(x)+m$, $p_{22}(x)=V(x)-m$ where m is the mass of the particle, the system (1.5) is known in relativistic quantum theory as a stationary one dimensional Dirac system or first canonical form of Dirac system [4].

We will use only one component $y_1(x,\lambda_n)$ of the two component vector eigenfunctions $y(x,\lambda_n)=[y_1(x,\lambda_n),y_2(x,\lambda_n)]^T$. $y(x,\lambda_n)$ be the eigenfunction corresponding to the eigenvalue λ_n of Dirac system (1.1)-(1.2). Suppose $x_n^{j,1}$ are the nodal points of the component $y_1(x,\lambda_n)$ of the eigenfunction $y(x,\lambda_n)$. In other words, $y_1(x_n^{j,1},\lambda_n)=0$. It is pointed out that the nodal points are repeated in accordance with the multiplicity of zeros. Let $I_n^{j,1}=\left(x_n^{j,1},x_n^{j+1,1}\right)$, and denote the nodal length by

$$l_n^{j,1}=x_n^{j+1,1}-x_n^{j,1}.$$

We also define the function $j_n(x)$ to be the largest index j such that $0\le x_n^{j,1}\le x$. Thus, $j=j_n(x)$ if and only if $x\in\left[x_n^{j,1},x_n^{j+1,1}\right]$ for $n>0$.

Denote $X=\{x_n^{j,1}\}$. X is called the set of nodal points for the component $y_1(x,\lambda_n)$ of the eigenfunction $y(x,\lambda_n)$ of the Dirac system.

Now, we will give some results which is given in [29] to use in main results.

Lemma 1.1. [29] *The spectrum of the problem (1.1)-(1.2) consists of eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}}$ which are all real and algebraically simple behave asymptotically as*

$$\lambda_n=n-2+\frac{v}{\pi}+\frac{c}{n}+O\left(\frac{1}{n^2}\right),n\to\infty, \tag{1.6}$$

and

$$\lambda_{-n}=-n+\frac{v}{\pi}-\frac{c}{n}+O\left(\frac{1}{n^2}\right),n\to\infty,$$

where

$$v=\int_0^\pi V(x)dx+\beta-\alpha, c=\frac{m^2}{2}+\frac{m}{2\pi}(\sin 2\alpha-\sin 2\beta)+\frac{a_0\sin\alpha-b_0\cos\alpha}{\pi}-\frac{a_1\sin\beta-b_1\cos\beta}{\pi}.$$

Lemma 1.2. [29] *Denote by* $y(x,\lambda)=\begin{pmatrix} y_1(x,\lambda) \\ y_2(x,\lambda) \end{pmatrix}$, *the solution of (1.1) satisfying the condition*

$$y(0,\lambda)=\begin{pmatrix} -(\lambda\sin\alpha+b_0) \\ \lambda\cos\alpha+a_0 \end{pmatrix}, \tag{1.7}$$

then, we have

$$\begin{aligned} y_1(x,\lambda) &= -\lambda\sin\left(\lambda x-\int_0^x V(t)dt+\alpha\right)+\frac{m^2}{2}x\cos\left(\lambda x-\int_0^x V(t)dt+\alpha\right) \\ &\quad -m\cos\alpha\sin\left(\lambda x-\int_0^x V(t)dt\right)-a_0\sin\left(\lambda x-\int_0^x V(t)dt\right) \\ &\quad -b_0\cos\left(\lambda x-\int_0^x V(t)dt\right)+O\left(\frac{e^{\tau x}}{\lambda}\right), \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} y_2(x,\lambda) &= -\lambda\cos\left(\lambda x-\int_0^x V(t)dt+\alpha\right)+\frac{m^2}{2}x\sin\left(\lambda x-\int_0^x V(t)dt+\alpha\right) \\ &\quad +m\sin\alpha\sin\left(\lambda x-\int_0^x V(t)dt\right)+a_0\cos\left(\lambda x-\int_0^x V(t)dt\right) \\ &\quad -b_0\sin\left(\lambda x-\int_0^x V(t)dt\right)+O\left(\frac{e^{\tau x}}{\lambda}\right), \end{aligned} \tag{1.9}$$

where $\tau=|\rho\lambda|$.

Lemma 1.3. [29] *For sufficiently large* $n>0$, *the first component* $y_1(x,\lambda_n)$ *of the eigenfunction* $y(x,\lambda_n)$ *of the Dirac system has exactly* $N(\alpha,\beta)$ *nodes in the interval* $(0,\pi)$ *where*

$$N(\alpha,\beta)=\begin{cases} n-2, & \text{for } \alpha\geq 0 \text{ and } \beta>0 \text{ or for } \alpha<0 \text{ and } \beta\leq 0 \\ n-3, & \text{for } \alpha\geq 0 \text{ and } \beta\leq 0 \\ n-1, & \text{for } \alpha<0 \text{ and } \beta>0 \end{cases}.$$

Moreover, uniformly with respect to $j\in\{1,2,...,N(\alpha,\beta)\}$, the nodal points and nodal lengths of the problem (1.1)-(1.2) satisfying the condition (1.7) has the following asymptotic formulas, respectively for sufficiently large n

$$x_n^{j,1}=\frac{j\pi+\int_0^{x_n^{j,1}} V(t)dt-\alpha}{\lambda_n}+\frac{m^2x_n^{j,1}}{2\lambda_n^2}+\frac{m\sin 2\alpha}{2\lambda_n^2}+\frac{a_0\sin\alpha-b_0\cos\alpha}{\lambda_n^2}+O\left(\frac{1}{\lambda_n^3}\right),$$

and

$$l_n^{j,1} = \frac{\pi}{\lambda_n} + \frac{1}{\lambda_n}\int\limits_{x_n^{j,1}}^{x_n^{j+1,1}} V(t)dt + \frac{m^2 l_n^{j,1}}{2\lambda_n^2} + O\left(\frac{1}{\lambda_n^3}\right),$$

or by using asymptotic formula

$$\lambda_n^{-1} = \frac{1}{n-2} - \frac{v}{(n-2)^2\pi} + \frac{v^2-\pi^2 c}{(n-2)^3\pi^2} + O\left(\frac{1}{n^4}\right),$$

we conclude

$$\begin{aligned} x_n^{j,1} \;&=\; \frac{j\pi + \int\limits_0^{x_n^{j,1}} V(t)dt - \alpha}{n-2} - \frac{\left(j\pi + \int\limits_0^{x_n^{j,1}} V(t)dt - \alpha\right) v}{(n-2)^2\pi} + \frac{m^2 x_n^{j,1} + m\sin 2\alpha + 2(a_0\sin\alpha - b_0\cos\alpha)}{2(n-2)^2} \\ &+ \frac{\left(j\pi + \int\limits_0^{x_n^{j,1}} V(t)dt - \alpha\right)\left(v^2-\pi^2 c\right)}{(n-2)^3\pi^2} - \frac{m^2 v x_n^{j,1} + mv\sin 2\alpha + 2v(a_0\sin\alpha - b_0\cos\alpha)}{(n-2)^3\pi^2} + O\left(\frac{1}{n^3}\right) \end{aligned}$$

and

$$\begin{aligned} l_n^{j,1} \;&=\; \frac{\pi + \int\limits_{x_n^{j,1}}^{x_n^{j+1,1}} V(t)dt}{n-2} - \frac{\left(\pi + \int\limits_{x_n^{j,1}}^{x_n^{j+1,1}} V(t)dt\right) v}{(n-2)^2\pi} + \frac{m^2\left(x_n^{j+1,1} - x_n^{j,1}\right)}{2(n-2)^2} \\ &+ \frac{\left(\pi + \int\limits_{x_n^{j,1}}^{x_n^{j+1,1}} V(t)dt\right)\left(v^2-\pi^2 c\right)}{(n-2)^3\pi^2} + O\left(\frac{1}{n^3}\right). \end{aligned}$$

Now, we will consider (1.1) equation with following boundary conditions

$$\begin{aligned} u_1(0)\cos\alpha + u_2(0)\sin\alpha &= 0, \\ u_1(\pi)\cos\beta + u_2(\pi)\sin\beta &= 0, \end{aligned} \tag{1.10}$$

where $0 \le \alpha, \beta \le \pi$, and m is positive in (1.3). It is well known that the spectrum of the problem (1.1) with the boundary conditions (1.10) consists of the eigenvalues $\lambda_n, n \in \mathbb{Z}$ which are all real and simple, and the sequence $\{\lambda_n, n \in \mathbb{Z}\}$ satisfies the classical asymptotic form [28]

$$\lambda_n = n + \frac{v}{\pi} + \frac{c_1}{n} + O\left(\frac{1}{n^2}\right), \tag{1.11}$$

where

$$v = \beta - \alpha + \int\limits_0^\pi V(x)dx, c_1 = \frac{m(\sin 2\alpha - \sin 2\beta) + m^2\pi}{2\pi\cos^2\left(\int\limits_0^\pi V(x)dx - \alpha + \beta\right)}.$$

Let $u(x,\lambda) = (u_1(x,\lambda), u_2(x,\lambda))^T$ be the solution of the (1.1) with the initial conditions

$$u_1(0,\lambda) = \sin\alpha, u_2(0,\lambda) = -\cos\alpha. \tag{1.12}$$

Then by the methods of successive approximations method [4], there hold

$$\begin{aligned} u_1(x,\lambda) &= \sin\left(\lambda x - \int\limits_0^x V(x)dx + \alpha\right) - \frac{U}{\lambda} + O\left(\frac{e^{|\tau\lambda|x}}{\lambda^2}\right), \\ u_2(x,\lambda) &= -\cos\left(\lambda x - \int\limits_0^x V(x)dx + \alpha\right) - \frac{V}{\lambda} + O\left(\frac{e^{|\tau\lambda|x}}{\lambda^2}\right) \end{aligned} \tag{1.13}$$

for large $|\lambda|$, where [28]

$$\begin{aligned} U(x,\lambda) &= -m\sin\left(\lambda x - \int\limits_0^x V(x)dx\right)\cos\alpha + \frac{m^2}{2}x\cos\left(\lambda x - \int\limits_0^x V(x)dx + \alpha\right), \\ V(x,\lambda) &= m\sin\left(\lambda x - \int\limits_0^x V(x)dx\right)\sin\alpha + \frac{m^2}{2}x\sin\left(\lambda x - \int\limits_0^x V(x)dx + \alpha\right). \end{aligned} \tag{1.14}$$

Lemma 1.4. [28] *For suffuciently large $|n|$, the $i-th$ component $u_i(x,\lambda_n)$ of the eigenfunction $u(x,\lambda_n)$ of the problem (1.1),(1.12) has exactly $|n|+1-i$ nodes in the interval $(0,\pi)$. Moreover, the asymptotic formulas for nodal points of first and second components of the eigenfunction $u(x,\lambda_n)$ as $n\to\infty$ uniformly with respect to $j\in\mathbb{Z}$ are as following*

$$x_n^{j,1} = \frac{j\pi}{\lambda_n} + \frac{1}{\lambda_n}\int\limits_0^{x_n^{j,1}} V(x)dx - \frac{\alpha}{\lambda_n} + \frac{(-1)^j m\sin 2\alpha}{2\lambda_n^2} + \frac{(-1)^j m^2 x_n^{j,1}}{2\lambda_n^2} + O\left(\frac{1}{\lambda_n^3}\right), \tag{1.15}$$

and

$$x_n^{j,2} = \frac{\left(j-\frac{1}{2}\right)\pi}{\lambda_n} + \frac{1}{\lambda_n}\int\limits_0^{x_n^{j,2}} V(x)dx - \frac{\alpha}{\lambda_n} + \frac{(-1)^{j+1} m\sin 2\alpha}{2\lambda_n^2} + \frac{(-1)^j m^2 x_n^{j,2}}{2\lambda_n^2} + O\left(\frac{1}{\lambda_n^3}\right). \tag{1.16}$$

The asymptotic expansions of nodal lengths $l_n^{j,1}$ are the following for the problem (1.1),(1.12) by using (1.15)

$$l_n^{j,1} = \frac{\pi}{\lambda_n} + \frac{1}{\lambda_n}\int\limits_{x_n^{j,1}}^{x_n^{j+1,1}} V(x)dx + \frac{\{(-1)^{j+1} - (-1)^j\}m\sin 2\alpha}{2\lambda_n^2} + \frac{\{(-1)^{j+1}x_n^{j+1,1} - (-1)^j x_n^{j,1}\}m^2}{2\lambda_n^2} + O\left(\frac{1}{\lambda_n^3}\right).$$

In case of $j=2k$(or $j=2k+1$), $k\in\mathbb{Z}$; we obtain

$$l_n^{j,1} = \frac{\pi}{\lambda_n} + \frac{1}{\lambda_n}\int\limits_{x_n^{j,1}}^{x_n^{j+1,1}} V(x)dx \pm \frac{m\sin 2\alpha}{2\lambda_n^2} \pm \frac{m^2\{x_n^{j+1,1} - x_n^{j,1}\}}{2\lambda_n^2} + O\left(\frac{1}{\lambda_n^3}\right). \tag{1.17}$$

We can obtain $l_n^{j,2}$ similarly as $|n| \to \infty$ by using definition of nodal lengths and (1.16).

Theorem 1.1. *Suppose that* $V \in L_1(0,\pi)$. Then, for almost every $x \in (0,\pi)$, with $j = j_n(x)$,

$$\lim_{n\to\infty} \lambda_n \int_{x_n^{j,i}}^{x_n^{j+1,i}} V(t)dt = V(x),$$

and

$$\lim_{n\to\infty} \lambda_n \int_{x_n^{j,i}}^{x_n^{j+1,i}} \cos(2\lambda_n \pi t)V(t)dt = 0,$$

where $i = 1, 2$ and

$$\lambda_n = \begin{cases} n-2, & \text{for the problem (1.1), (1.7),} \\ n, & \text{for the problem (1.1), (1.12).} \end{cases}$$

Proof: We prove this theorem by considering similar method in [1]. It is well known

$$\lim_{n\to\infty} \frac{s_n}{\pi} \int_{x_n^{j,i}}^{x_n^{j+1,i}} V(t)dt = V(x) \text{ and } \quad \lim_{n\to\infty} \frac{s_n}{\pi} \int_{x_n^{j,i}}^{x_n^{j+1,i}} \cos(2s_n t)V(t)dt = 0$$

for almost every $x \in (0,\pi)$ when n is sufficiently large [30]. If we use the fact

$$\lim_{n\to\infty} \frac{s_n}{\pi\lambda_n} = 1,$$

we get

$$\begin{aligned} \lim_{n\to\infty} \frac{s_n}{\pi} \int_{x_n^{j,i}}^{x_n^{j+1,i}} V(t)dt &= \lim_{n\to\infty} \lambda_n \int_{x_n^{j,i}}^{x_n^{j+1,i}} V(t)dt . \lim_{n\to\infty} \frac{s_n}{\pi\lambda_n} \\ &= \lim_{n\to\infty} \lambda_n \int_{x_n^{j,i}}^{x_n^{j+1,i}} V(t)dt = V(x), \end{aligned}$$

and

$$\begin{aligned} \lim_{n\to\infty} \frac{s_n}{\pi} \int_{x_n^{j,i}}^{x_n^{j+1,i}} \cos(2s_n t)V(t)dt &= \lim_{n\to\infty} \lambda_n \int_{x_n^{j,i}}^{x_n^{j+1,i}} \cos(2\lambda_n \pi t)V(t)dt \lim_{n\to\infty} \frac{s_n}{\pi\lambda_n} \\ &= \lim_{n\to\infty} \lambda_n \int_{x_n^{j,i}}^{x_n^{j+1,i}} \cos(2\lambda_n \pi t)V(t)dt = 0. \end{aligned}$$

The above equalities are still valid even if $x_n^{j,1}$ and $x_n^{j+1,1}$ are replaced by $X_n^{j,1}$ and $X_n^{j+1,1}$.

2. Reconstruction of Potential function by using nodal points

In this section, we will derive some reconstruction formulas of potential functions for the problems (1.1),(1.7) and (1.1), (1.12). Here and later, we denote L_k^n (the grid lengths) to go with X_k^n in the same way that l_k^n (the nodal lengths) to go with x_k^n [1].

Theorem 2.1. *Let $V \in L_1[0,\pi]$ be the potential function for Dirac system (1.1). Define F_n by*

a) For the problem (1.1)-(1.7),

$$F_n = (n-2)\left\{\sum_{k=1}^{n-1}\left[n-2-\frac{m^2}{2(n-2)}\right]L_k^{(n)}-\pi\right\}.$$

b) For the problem (1.1)-(1.12),

$$F_n = n\left\{\sum_{k=1}^{n-1}\left[n\pm\frac{m^2}{2n}\right]L_k^{(n)}\pm\frac{m^2}{n}X_k^{(n)}-\pi\right\}.$$

Then, F_n converges to V pointwisely almost everywhere and also in L_1 sense. Furthermore, pointwise convergence holds for all the continuity points of V.

Proof:

a) We will consider the reconstruction formula for the potential functions for the problem (1.1)-(1.7). Observe that, by Lemma 1.3, we have

$$\frac{\lambda_n}{\pi}l_n^{j,1}-1-\frac{m^2 l_n^{j,1}}{2\lambda_n\pi}=\frac{1}{\pi}\int_{x_n^{j,1}}^{x_n^{j+1,1}}V(x)dx+O\left(\frac{1}{\lambda_n^2}\right),$$

and

$$\lambda_n\left[l_n^{j,1}\left(\lambda_n-\frac{m^2}{2\lambda_n}\right)-\pi\right]=\lambda_n\int_{x_n^{j,1}}^{x_n^{j+1,1}}V(x)dx+O\left(\frac{1}{\lambda_n}\right).$$

Then, by using asymptotic expansions for eigenvalues below, we obtain

$$\begin{aligned}\lambda_n\left[l_n^{j,1}\left(\lambda_n-\frac{m^2}{2\lambda_n}\right)-\pi\right] &= \left[n-2+O\left(\frac{1}{n}\right)\right]\left[l_n^{j,1}\left(n-2+O\left(\frac{1}{n}\right)-\frac{m^2}{2(n-2)+O\left(\frac{1}{n}\right)}\right)-\pi\right]\\ &= \left[n-2+O\left(\frac{1}{n}\right)\right]\left[l_n^{j,1}\left(n-2-\frac{m^2}{2(n-2)}+O\left(\frac{1}{n}\right)\right)-\pi\right]\\ &= (n-2)\left[l_n^{j,1}\left(n-2-\frac{m^2}{2(n-2)}\right)-\pi\right]+o(1).\end{aligned}$$

Hence, to prove Theorem 2.1 (a), it suffices to show Theorem 2.2.

(b) it can be proved similarly. To complete the proof of Theorem 2.1. (b), it suffices to show Theorem 2.3.

Theorem 2.2. *The potential function $V \in L_1(0,\pi)$ of the problem (1.1),(1.7) satisfies*

$$V(x) = \lim_{n\to\infty} \left[l_n^{j,1}\lambda_n - m^2 \frac{l_n^{j,1}}{2\lambda_n} - \pi \right] \lambda_n,$$

for almost every $x \in (0,\pi)$, with $j = j_n(x)$.

Proof. By Lemma 1.3,

$$l_n^{j,1} - \frac{m^2 l_n^{j,1}}{2\lambda_n^2} = \frac{\pi}{\lambda_n} + \frac{\int\limits_{x_n^{j,1}}^{x_n^{j+1,1}} V(x)dx}{\lambda_n} + O\left(\frac{1}{\lambda_n^3}\right),$$

so that

$$\left[\left(l_n^{j,1} - \frac{m^2 l_n^{j,1}}{2\lambda_n^2} \right) \lambda_n - \pi \right] \lambda_n = \lambda_n \int\limits_{x_n^{j,1}}^{x_n^{j+1,1}} V(x)dx + O\left(\frac{1}{\lambda_n}\right). \tag{2.1}$$

We may assume $x_n^{j,1} \neq x$. By Theorem 1.1., if we take limit both sides of (2.1) as $n \to \infty$ for almost $x \in (0,\pi)$, we get

$$V(x) = \lim_{n\to\infty} \left[l_n^{j,1}\lambda_n - m^2 \frac{l_n^{j,1}}{2\lambda_n} - \pi \right] \lambda_n.$$

Theorem 2.3. *The potential function $V \in L_1(0,\pi)$ of the problem (1.1),(1.12) satisfies*

$$V(x) = \lim_{n\to\infty} \left[l_n^{j,1}\lambda_n \pm m^2 \frac{(x_n^{j,1} + x_n^{j+1,1})}{2\lambda_n} - \pi \right] \lambda_n \pm m \sin 2\alpha,$$

for almost every $x \in (0,\pi)$, with $j = j_n(x)$.

Proof: It can be proved by similar process to Theorem 2.2.

3. Main Results about Lipschitz Stability

In this section, we solve a Lipschitz stability problem for Dirac operator. Lipschitz stability is about a continuity between two metric spaces. So, we have to first reconstruct to metric spaces. To show this continuity, we will use a homeomorphism between these spaces. Stability problems were studied by many authors [2], [31], [32] . We will give a main theorem which execute that the inverse nodal problem for Dirac equation is stable with Lipschitz stability.

We denote the space Ω_{Dir} as a collection of all Dirac operators and the space Σ_{Dir} as a collection of all admissible double sequences of nodes such that corresponding functions are convergent in L_1. A pseudometric $d_{\Sigma_{Dir}}$ on Σ_{Dir} will be defined. Essentialy, $d_{\Sigma_{Dir}}(X,\overline{X})$ is so close to

$$d_0(X,\overline{X}) = \overline{\lim_{n\to\infty}}\, \pi \left[n - 2 - \frac{m^2}{2(n-2)} \right] \sum_{k=1}^{n-1} \left| L_k^n - \overline{L}_k^n \right|,$$

where $n \geq \frac{m}{\sqrt{2}} + 2$ and $L_k^{(n)} = X_{k+1}^{(n)} - X_k^{(n)}$, $\overline{L}_k^{(n)} = \overline{X}_{k+1}^{(n)} - \overline{X}_k^{(n)}$.

If we define $X \sim \overline{X}$ if and only if $d_{\Sigma_{Dir}}(X, \overline{X}) = 0$, then $\sim$ is an equivalence relation on Σ_{Dir} and $d_{\Sigma_{Dir}}$ would be a metric for the partition set $\Sigma^*_{Dir} = \Sigma_{Dir}/\sim$. Let $\Sigma_{Dir_1} \subset \Sigma_{Dir}$ be the subspace of all asymptotically equivalent nodal sequences and let $\Sigma^*_{Dir_1} = \Sigma_{Dir_1}/\sim$. Let Φ be a homeomorphism the maps Ω_{Dir} onto $\Sigma^*_{Dir_1}$. We will call Φ a nodal map.

Definition 3.1. Let $\mathbb{N}' = \mathbb{N} - \{1\}$. We denote

(i)

$$\begin{aligned} \Omega_{Dir} &= \{V \in L_1(0,\pi) : V(x) \textit{ is the potential function of the Dirac equation}\}, \\ \Sigma_{Dir} &= \textit{The collection of the all double sequences defined as} \end{aligned}$$

$$X = \left\{X_k^n : k = 1, 2, ..., n; n \in \mathbb{N}', 0 < X_1^n < X_2^n < ... < X_{n-1}^n < \pi\right\},$$

for each $n \in \mathbb{N}$.

(ii) Let $X \in \Sigma_{Dir}$ and define $X = \{X_k^n\}$ where $L_k^n = X_{k+1}^n - X_k^n$ and $I_k^n = \left(X_k^n, X_{k+1}^n\right)$. We say X is quasinodal to some $V \in L_1(0,\pi)$ if X is an admissible sequence of nodes and satisfies (I) and (II) below:

(I) X has the following asymptotics uniformly for k, as $n \to \infty$

$$X_k^n = \frac{k\pi}{n-2} + O\left(\frac{1}{n}\right), k = 1, 2, ..., n$$

for the problem (1.1),(1.7). And the sequence of the functions

$$F_n = (n-2)\left\{\sum_{k=1}^{n-1}\left[n - 2 - \frac{m^2}{2(n-2)}\right]L_k^{(n)} - \pi\right\},$$

converges to V in L_1.

(II) For the problem (1.1),(1.12), X has the following asymptotics uniformly for k, as $n \to \infty$

$$X_k^n = \frac{k\pi}{n} + O\left(\frac{1}{n}\right), k = 1, 2, ..., n$$

And the sequence of the functions

$$F_n = n\left\{\sum_{k=1}^{n-1}\left[n \pm \frac{m^2}{2n}\right]L_k^{(n)} \pm \frac{m^2}{n}X_k^{(n)} - \pi\right\},$$

converges to V in L_1.

Lemma 3.1. Let $X, X \in \Sigma_{Dir}$.

a) If X belongs to case I, then

$$L_k^n = \frac{\pi}{n-2} + O\left(\frac{1}{n}\right), k = 1, 2, ..., n$$

If X belongs to case II, then

$$L_k^n = \frac{\pi}{n} + O\left(\frac{1}{n}\right), k = 1, 2, ..., n$$

b) $\chi_{n,k} = \left|X_k^n - \overline{X}_k^n\right| = O\left(\frac{1}{n}\right).$

c) For all $x \in (0, \pi)$*, define* $J_n(x) = maks\{k : X_k^n \leq x\}$ *so that* $k = J_n(x)$ *if and only if* $x \in [X_J^n, X_{J+1}^n]$*. Then, for sufficiently large* n*,*

$$\left|J_n(x) - \overline{J}_n(x)\right| \leq 1.$$

Proof:

a) For case I, we get

$$L_k^n = X_{k+1}^n - X_k^n = \frac{(k+1)\pi}{n-2} - \frac{k\pi}{n-2} + O\left(\frac{1}{n}\right) = \frac{\pi}{n-2} + O\left(\frac{1}{n}\right),$$

by using the definition of nodal lengths, and, similarly for the case (II), we obtain

$$L_k^n = X_{k+1}^n - X_k^n = \frac{(k+1)\pi}{n} - \frac{k\pi}{n} + O\left(\frac{1}{n}\right) = \frac{\pi}{n} + O\left(\frac{1}{n}\right).$$

b) We will only consider case I. The other case is similar. By the asymptotic estimates

$$\begin{aligned}
\left|\chi_{n,k}\right| &= \left|X_k^n - \overline{X}_k^n\right| = \left|X_k^n - \frac{k\pi}{n-2} + \frac{k\pi}{n-2} - \overline{X}_k^n\right| \\
&\leq \left|X_k^n - \frac{k\pi}{n-2}\right| + \left|\frac{k\pi}{n-2} - \overline{X}_k^n\right| \\
&= O\left(\frac{1}{n}\right) + O\left(\frac{1}{n}\right) \\
&= O\left(\frac{1}{n}\right).
\end{aligned}$$

c) Fix $x \in (0, \pi)$. Let $J = J_n(x)$ and $\overline{J} = \overline{J}_n(x)$. Since

$$X_J^n \leq x \leq X_{J+1}^n \Rightarrow \frac{J\pi}{n-2} + O\left(\frac{1}{n}\right) = X_J^{(n)} \leq x \leq X_{J+1}^{(n)} = \frac{(J+1)\pi}{n-2} + O\left(\frac{1}{n}\right),$$

and

$$\overline{X}_{\overline{J}}^n \leq x \leq \overline{X}_{\overline{J}+1}^n \Rightarrow \frac{\overline{J}\pi}{n-2} + O\left(\frac{1}{n}\right) = \overline{X}_{\overline{J}}^{(n)} \leq x \leq \overline{X}_{\overline{J}+1}^{(n)} = \frac{\overline{J}\pi}{n-2} + O\left(\frac{1}{n}\right),$$

when n is large enough, $\overline{J} + 1 \geq J$ and $J + 1 \geq \overline{J}$. Hence, $-1 \leq \overline{J} - J \leq 1$, then $\left|\overline{J} - J\right| \leq 1$.

Definition 3.2. Suppose that $X, \overline{X} \in \Sigma_{Dir}$ with L_k^n and $\overline{L}_k^n$ as their respective grid lengths. Let

$$S_n\left(X, \overline{X}\right) = \pi\left[n - 2 - \frac{m^2}{2(n-2)}\right] \sum_{k=1}^{n-1} \left|L_k^n - \overline{L}_k^n\right|. \tag{3.1}$$

Define

$$d_0\left(X,\overline{X}\right)=\overline{\lim_{n\to\infty}}S_n\left(X,\overline{X}\right)\ \text{and}\ d_{\Sigma_{Dir}}\left(X,\overline{X}\right)=\overline{\lim_{n\to\infty}}\frac{S_n\left(X,\overline{X}\right)}{1+S_n\left(X,\overline{X}\right)}.$$

We obtain this metric by evaluating $\left\|V-\overline{V}\right\|_1$ in Theorem 3.1. This definition was first made by [1]. Since the function $f(x)=\dfrac{x}{1+x}$ is monotonic, we have

$$d_{\Sigma_{Dir}}\left(X,\overline{X}\right)=\frac{d_0\left(X,\overline{X}\right)}{1+d_0\left(X,\overline{X}\right)}\in[0,\pi],$$

admitting that if $d_0\left(X,\overline{X}\right)=\infty$, then $d_{\Sigma_{Dir}}\left(X,\overline{X}\right)=1$. Conversely

$$d_0\left(X,\overline{X}\right)=\frac{d_{\Sigma_{Dir}}\left(X,\overline{X}\right)}{1-d_{\Sigma_{Dir}}\left(X,\overline{X}\right)}.$$

We can easily proof this equality by using Law's method [1], [2].

Lemma 3.2. *Let $X,\overline{X}\in\Sigma_{Dir}$.*
a) $d_{\Sigma_{Dir}}$ is a pseudometric on Σ_{Dir}.
b) If X and $\overline{X}$ belong to different cases, then $d_{\Sigma_{Dir}}\left(X,\overline{X}\right)=1$.

Proof: It can be proved similar way with [1].

After following therem, we can say that the inverse nodal problem for the Dirac operator is Lipschitz stable.

Theorem 3.1. *The metric spaces $(\Omega_{Dir},\|.\|_1)$ and $\left(\Sigma^*_{Dir_1}/\sim,d_{\Sigma_{Dir}}\right)$ are homeomorphic to each other. Here $\sim$ is the equivalence relation induced by $d_{\Sigma_{Dir}}$. Furthermore*

$$\left\|V-\overline{V}\right\|_1=\frac{d_{\Sigma_{Dir}}\left(X,\overline{X}\right)}{1-d_{\Sigma_{Dir}}\left(X,\overline{X}\right)},$$

where $d_{\Sigma_{Dir}}\left(X,\overline{X}\right)<1$.

Proof: In view of Lemma 3.2. we only need to consider when $X,\overline{X}\in\Sigma_{Dir}$ belong to same case. Without loss of generality, let $X,\overline{X}$ belong to case I. In this case, we shall show that

$$\left\|V-\overline{V}\right\|_1=d_0\left(X,\overline{X}\right).$$

According to the Theorem 2.1., F_n and $\overline{F}_n$ convergence to V and $\overline{V}$, respectively. If we use the definition of norm in L_1 for the functions V and $\overline{V}$, we have

$$\left\|V-\overline{V}\right\|_1=(n-2)\left[n-2-\frac{m^2}{2(n-2)}\right]\int_0^{\pi}\left|L^{(n)}_{J_n(x)}-\overline{L}^{(n)}_{\overline{J}_n(x)}\right|dx+o(1).$$

Hence by Fatou's Lemma,

$$\begin{aligned}\left\|V-\overline{V}\right\|_1 &\leq (n-2)\left[n-2-\frac{m^2}{2(n-2)}\right]\int_0^\pi \left|L_{J_n(x)}^{(n)}-\overline{L}_{J_n(x)}^{(n)}\right| dx \\ &\quad +(n-2)\left[n-2-\frac{m^2}{2(n-2)}\right]\int_0^\pi \left|\overline{L}_{J_n(x)}^{(n)}-\overline{L}_{\overline{J}_n(x)}^{(n)}\right| dx. \end{aligned}\tag{3.2}$$

Here, second and first term can be written as

$$\int_0^\pi \left|\overline{L}_{J_n(x)}^{(n)}-\overline{L}_{\overline{J}_n(x)}^{(n)}\right| dx = o\left(\frac{1}{n^3}\right),$$

and

$$\int_0^\pi \left|L_{J_n(x)}^{(n)}-\overline{L}_{J_n(x)}^{(n)}\right| dx = \frac{\pi}{n-2}\sum_{k=1}^{n-1}\left|L_k^n-\overline{L}_k^n\right| + o\left(\frac{1}{n^2}\right).$$

If we consider last equalities in (3.2), we get

$$\begin{aligned}\left\|V-\overline{V}\right\|_1 &\leq (n-2)\left[n-2-\frac{m^2}{2(n-2)}\right] o\left(\frac{1}{n^3}\right) \\ &\quad +(n-2)\left[n-2-\frac{m^2}{2(n-2)}\right]\left[\frac{\pi}{n-2}\sum_{k=1}^{n-1}\left|L_k^n-\overline{L}_k^n\right| + o\left(\frac{1}{n^2}\right)\right],\end{aligned}$$

and

$$\left\|V-\overline{V}\right\|_1 \leq \pi\left[n-2-\frac{m^2}{2(n-2)}\right]\sum_{k=1}^{n-1}\left|L_k^n-\overline{L}_k^n\right| + o(1). \tag{3.3}$$

Similarly, we can get

$$\left\|V-\overline{V}\right\|_1 \geq \pi\left[n-2-\frac{m^2}{2(n-2)}\right]\sum_{k=1}^{n-1}\left|L_k^n-\overline{L}_k^n\right| + o(1). \tag{3.4}$$

If we consider (3.3) and (3.4) together, we obtain

$$\left\|V-\overline{V}\right\|_1 = \pi\left[n-2-\frac{m^2}{2(n-2)}\right]\sum_{k=1}^{n-1}\left|L_k^n-\overline{L}_k^n\right|.$$

The proof is complete after taking the limit as $n\to\infty$.

3. Conclusion

In this study, it was given a Lipschtiz stability of inverse nodal problem for Dirac operator by using zeros of one component of two component vector eigenfunctions. Especially, two metric spaces was defined and it was shown that these two metric spaces were homeomorphic to each other. These results are new and can be generalized.

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